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Integrable Couplings of the Generalized AKNS Hierarchy with an Arbitrary Function and Its Bi-Hamiltonian Structure

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Abstract We construct a new loop algebra $\widetilde{A_3}$, which is used to set up an isospectral problem. Then a new integrable couplings of the generalized AKNS hierarchy is derived, which possesses bi-Hamiltonian structure and contains an arbitrary spatial function. As its reduction, we gain the integrable couplings of the Schrödinger equation. Furthermore, many conserved quantities of the integrable couplings are obtained.

Keywords Loop algebra · Integrable couplings · Hamiltonian structure

1 Introduction

Finding new Liouville integrable systems and their Hamiltonian structures is a central and difficult topic in soliton theory [1–4]. Bi-Hamiltonian formulation is significant for investigating integrable properties of nonlinear systems of differential equations [5]. Many mathematical and physical systems have been found to possess bi-Hamiltonian structure [6–11]. In addition, the study of integrable couplings of soliton equations has attracted much attention recently. It originated from the investigations into the symmetry problems and associated centreless Virasoro algebras [12]. The problem of integrable couplings can be expressed as follows: For a given integrable system, how can we construct a non-trivial system of differential equations which is still integrable and includes the original integrable system as a sub-system? Note that a change of orders of dependent variables does not lose integrability. Therefore, up to a permutation, for a given integrable system of evolution type $u_t = K(u)$, we actually need to construct a new bigger integrable system as follows:

$$\begin{cases} u_t = K(u), \\ v_t = S(u, v). \end{cases}$$

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The vector-valued function S should satisfy the non-triviality condition $\frac{\partial S}{\partial [u]} \neq 0$, where $[u]$ denotes a vector consisting of all derivatives of u with respect to the space variable. A few ways to construct integrable couplings of soliton equations are presented by perturbation, enlarging spectral problems, constructing new loop Lie algebra and creating semidirect sums of Lie algebra [13, 14]. Many Integrable couplings systems of the well-known integrable hierarchies have been worked out such as AKNS hierarchy, Toda hierarchy, JM hierarchy, KN hierarchy and so on.

In this paper, we present a new linear isospectral problem based on a new loop algebra $\widetilde{A_3}$. A new integrable couplings with an arbitrary spatial function of the generalized AKNS hierarchy is generated, which is integrable in Liouville sense. Then its bi-Hamiltonian structure is obtained by making use of the trace variational identity [15] and the quadratic-form identity [16]. Finally, we obtain many conserved quantities of the integrable couplings.

2 Integrable Couplings of the Generalized AKNS Hierarchy with an Arbitrary Function

We consider Lie algebra A_3 as follows:

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$
$$e_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1}$$

$$\begin{gathered} [e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1, \quad [e_1, e_5] = 2e_5, \\ [e_1, e_6] = -2e_6, \quad [e_2, e_4] = -2e_5, \quad [e_2, e_6] = e_4, \quad [e_3, e_4] = 2e_6, \\ [e_3, e_5] = -e_4. \end{gathered} \tag{2}$$

In terms of Lie algebra A_3, a new loop algebra $\widetilde{A_3}$ is given by

$$\widetilde{A_3} = \operatorname{span}\{e_k(i,n), i = 0, 1\}_{k=1}^{6}, e_k(i,n) = e_k \lambda^{2n+i}, \quad i = 0, 1, \tag{3}$$

along with the commutative operations

$$[e_1(i,m), e_3(j,n)] = \begin{cases} -2e_3(i+j, m+n), & i+j < 2, \\ -2e_3(i+j-2, m+n+1), & i+j \geq 2, \end{cases}$$
$$[e_1(i,m), e_2(j,n)] = \begin{cases} 2e_2(i+j, m+n), & i+j < 2, \\ 2e_2(i+j-2, m+n+1), & i+j \geq 2, \end{cases}$$
$$[e_2(i,m), e_3(j,n)] = \begin{cases} e_1(i+j, m+n), & i+j < 2, \\ e_1(i+j-2, m+n+1), & i+j \geq 2, \end{cases}$$

$$[e_1(i,m), e_5(j,n)] = \begin{cases} 2e_5(i+j, m+n), & i+j<2, \\ 2e_5(i+j-2, m+n+1), & i+j \geq 2, \end{cases}$$

$$[e_1(i,m), e_6(j,n)] = \begin{cases} -2e_6(i+j, m+n), & i+j<2, \\ -2e_6(i+j-2, m+n+1), & i+j \geq 2, \end{cases} \tag{4}$$

$$[e_2(i,m), e_4(j,n)] = \begin{cases} -2e_5(i+j, m+n), & i+j<2, \\ -2e_5(i+j-2, m+n+1), & i+j \geq 2, \end{cases}$$

$$[e_2(i,m), e_6(j,n)] = \begin{cases} e_4(i+j, m+n), & i+j<2, \\ e_4(i+j-2, m+n+1), & i+j \geq 2, \end{cases}$$

$$[e_3(i,m), e_5(j,n)] = \begin{cases} -e_4(i+j, m+n), & i+j<2, \\ -e_4(i+j-2, m+n+1), & i+j \geq 2, \end{cases}$$

$$[e_3(i,m), e_4(j,n)] = \begin{cases} 2e_6(i+j, m+n), & i+j<2, \\ 2e_6(i+j-2, m+n+1), & i+j \geq 2, \end{cases}$$

$$\deg(e_k(i,n)) = 2n+i, \qquad k=1,2,\ldots,6,\ i=0,1.$$

Set

$$\widetilde{A_{31}} = \text{span}\{e_1(i,n), e_2(i,n), e_3(i,n)\},$$
$$\widetilde{A_{32}} = \text{span}\{e_4(i,n), e_5(i,n), e_6(i,n)\},$$

we find that

$$\widetilde{A_3} = \widetilde{A_{31}} \oplus \widetilde{A_{32}},\ [\widetilde{A_{31}}, \widetilde{A_{32}}] \subset \widetilde{A_{32}}. \tag{5}$$

In terms of $\widetilde{A_3}$, consider an isospectral problem

$$\begin{aligned} &\phi_x = U\phi, \qquad \phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T, \\ &U = e_1(1,0) + u_1 e_2(0,0) + u_2 e_3(0,0) + p(x)e_4(0,0) + u_3 e_5(0,0) + u_4 e_6(0,0) \\ &\quad = \begin{pmatrix} \lambda & u_1 & p(x) & u_3 \\ u_2 & -\lambda & u_4 & -p(x) \\ 0 & 0 & \lambda & u_1 \\ 0 & 0 & u_2 & -\lambda \end{pmatrix}, \end{aligned} \tag{6}$$

where $p(x)$ is an arbitrary differentiable function of x.

Set

$$\begin{aligned} V = \sum_{m=0}^{\infty}\sum_{i=0}^{1} &(a(i,m)e_1(i,-m) + b(i,m)e_2(i,-m) + c(i,m)e_3(i,-m) + d(i,m)e_4(i,-m) \\ &+ f(i,m)e_5(i,-m) + g(i,m)e_6(i,-m)). \end{aligned}$$

Solving the stationary zero-curvature equation

$$V_x = [U, V] \tag{7}$$

yields

$$a_x(0,m) = u_1 c(0,m) - u_2 b(0,m), \qquad a_x(1,m) = u_1 c(1,m) - u_2 b(1,m),$$

$$
\begin{aligned}
&b_x(0,m)=2b(1,m+1)-2u_1a(0,m), \qquad b_x(1,m)=2b(0,m)-2u_1a(1,m),\\
&c_x(0,m)=-2c(1,m+1)+2u_2a(0,m), \qquad c_x(1,m)=-2c(0,m)+2u_2a(1,m),\\
&d_x(0,m)=u_1g(0,m)-u_2f(0,m)+u_3c(0,m)-u_4b(0,m),\\
&d_x(1,m)=u_1g(1,m)-u_2f(1,m)+u_3c(1,m)-u_4b(1,m),\\
&f_x(0,m)=2f(1,m+1)-2u_1d(0,m)-2u_3a(0,m)+2p(x)b(0,m),\\
&f_x(1,m)=2f(0,m)-2u_1d(1,m)-2u_3a(1,m)+2p(x)b(1,m),\\
&g_x(0,m)=-2g(1,m+1)+2u_2d(0,m)+2u_4a(0,m)-2p(x)c(0,m),\\
&g_x(1,m)=-2g(0,m)+2u_2d(1,m)+2u_4a(1,m)-2p(x)c(1,m),\\
&a(0,0)=\beta, \qquad a(1,0)=b(0,0)=0,\\
&b(1,0)=c(0,0)=c(1,0)=d(0,0)=d(1,0)=f(0,0)=f(1,0)=g(0,0)\\
&\quad =g(1,0)=0,\\
&a(0,1)=-\frac{1}{2}\beta u_1u_2, a(1,1)=0, b(0,1)=\frac{1}{2}\beta u_{1x}, b(1,1)=\beta u_1, c(1,1)=\beta u_2,\\
&f(1,1)=\beta u_3,\\
&g(1,1)=\beta u_4, \qquad d(1,1)=0, \qquad f(0,1)=\frac{1}{2}\beta u_{3x}-\beta u_1p(x),\\
&g(0,1)=-\frac{1}{2}\beta u_{4x}-\beta u_2p(x), \quad \ldots.
\end{aligned} \tag{8}
$$

Note

$$
\begin{aligned}
V_+^{(n)} &= \sum_{m=0}^{n}\sum_{i=0}^{1}(a(i,m)e_1(i,n-m)+b(i,m)e_2(i,n-m)+c(i,m)e_3(i,n-m)\\
&\quad + d(i,m)e_4(i,n-m)+f(i,m)e_5(i,n-m)+g(i,m)e_6(i,n-m)),\\
V_-^{(n)} &= \lambda^{2n}V-V_+^{(n)},
\end{aligned}
$$

then we have

$$
-V_{+x}^{(n)}+[U,V_+^{(n)}]=V_{-x}^{(n)}-[U,V_-^{(n)}]. \tag{9}
$$

It is easy to verify that the terms of the left-hand side in (9) are of the degree ≥ 0, while the terms of the right-hand side in (9) are of degree ≤ 0. Therefore, the terms of both sides in (9) are of degree 0. It follows that

$$
\begin{aligned}
-V_{+x}^{(n)}+[U,V_+^{(n)}] &= -2b(1,n+1)e_2(0,0)+2c(1,n+1)e_3(0,0)-2f(1,n+1)e_5(0,0)\\
&\quad +2g(1,n+1)e_6(0,0).
\end{aligned}
$$

Taking $V^{(n)}=V_+^{(n)}$, then the zero-curvature equation

$$
U_t-V_x^{(n)}+[U,V^{(n)}]=0 \tag{10}
$$

admits the Lax integrable system

$$u_t = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \\ -2 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} c(1,n+1)+g(1,n+1) \\ b(1,n+1)+f(1,n+1) \\ c(1,n+1) \\ b(1,n+1) \end{pmatrix}$$

$$= J_1 \begin{pmatrix} c(1,n+1)+g(1,n+1) \\ b(1,n+1)+f(1,n+1) \\ c(1,n+1) \\ b(1,n+1) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 2u_1\partial^{-1}u_1 & \partial - 2u_1\partial^{-1}u_2 \\ 0 & 0 & \partial - 2u_2\partial^{-1}u_1 & 2u_2\partial^{-1}u_2 \\ 2u_1\partial^{-1}u_1 & \partial - u_1\partial^{-1}u_2 & J_{233} & J_{234} \\ \partial - 2u_2\partial^{-1}u_1 & 2u_2\partial^{-1}u_2 & J_{234} & J_{244} \end{pmatrix}$$

$$\times \begin{pmatrix} c(0,n)+g(0,n) \\ b(0,n)+f(0,n) \\ c(0,n) \\ b(0,n) \end{pmatrix} = J_2 \begin{pmatrix} c(0,n)+g(0,n) \\ b(0,n)+f(0,n) \\ c(0,n) \\ b(0,n) \end{pmatrix}. \tag{11}$$

Where

$$J_{233} = 2u_1\partial^{-1}u_3 + 2u_3\partial^{-1}u_1 - 2u_1\partial^{-1}u_1,$$

$$J_{234} = -2u_1\partial^{-1}u_4 - 2u_3\partial^{-1}u_2 + 2u_1\partial^{-1}u_2 - \partial - 2p(x),$$

$$J_{243} = -2u_2\partial^{-1}u_3 - 2u_4\partial^{-1}u_1 + 2u_2\partial^{-1}u_1 - \partial + 2p(x),$$

$$J_{244} = 2u_2\partial^{-1}u_4 + 2u_4\partial^{-1}u_2 - 2u_2\partial^{-1}u_2,$$

and J_1, J_2 are all Hamiltonian operators.

3 The Bi-Hamiltonian Structure

Let $R^6 = \{X = (x_1, \dots, x_6)^T | x_i \in R, i = 1, \dots, 6\}$, for $\forall a = (a_1, a_2, \dots, a_6)^T$, $b = (b_1, b_2, \dots, b_6)^T \in R^6$, define a commutator

$$[a, b]^T = (a_1, a_2, \dots, a_6)R(b) = a^T R(b), \tag{12}$$

where

$$R(b) = \begin{pmatrix} 0 & 2b_2 & -2b_3 & 0 & 2b_5 & -2b_6 \\ b_3 & -2b_1 & 0 & b_6 & -2b_4 & 0 \\ -b_2 & 0 & 2b_1 & -b_5 & 0 & 2b_4 \\ 0 & 0 & 0 & 0 & 2b_2 & -2b_3 \\ 0 & 0 & 0 & b_3 & -2b_1 & 0 \\ 0 & 0 & 0 & -b_2 & 0 & 2b_1 \end{pmatrix},$$

then R^6 is a Lie algebra with (12).

In terms of R^6, an isospectral problem is taken as

$$\varphi_x = [\tilde{U}, \varphi], \; \tilde{U} = (\lambda, u_1, u_2, p(x), u_3, u_4)^T, \; \varphi = (\varphi_1, \dots, \varphi_6)^T, \tag{13}$$

$$\varphi_t = [\tilde{V}^{(n)}, \varphi],$$

$$\tilde{V}^{(n)} = \begin{pmatrix} \sum_{m=0}^{n}(a(0,m) + \lambda a(1,m))\lambda^{2(n-m)} \\ \sum_{m=0}^{n}(b(0,m) + \lambda b(1,m))\lambda^{2(n-m)} \\ \sum_{m=0}^{n}(c(0,m) + \lambda c(1,m))\lambda^{2(n-m)} \\ \sum_{m=0}^{n}(d(0,m) + \lambda d(1,m))\lambda^{2(n-m)} \\ \sum_{m=0}^{n}(f(0,m) + \lambda f(1,m))\lambda^{2(n-m)} \\ \sum_{m=0}^{n}(g(0,m) + \lambda g(1,m))\lambda^{2(n-m)} \end{pmatrix}, \tag{14}$$

then the zero curvature equation

$$\tilde{U}_t - \tilde{V}_x^{(n)} + [\tilde{U}, \tilde{V}^{(n)}] = 0 \tag{15}$$

also generates the integrable hierarchy (11).

Solving the matrix equation for F

$$R(b)F = -(R(b)F)^T$$

gives

$$F = \begin{pmatrix} 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{16}$$

Define a linear function

$$\{a, b\} = a^T F b = (2a_1 + 2a_4)b_1 + (a_3 + a_6)b_2 + (a_2 + a_5)b_3 + 2a_1 b_4 + a_3 b_5 + a_2 b_6,$$

a direct calculation gives

$$\left\{\tilde{V}, \frac{\partial \tilde{U}}{\partial u_1}\right\} = c(0) + \lambda c(1) + g(0) + \lambda g(1),$$

$$\left\{\tilde{V}, \frac{\partial \tilde{U}}{\partial u_2}\right\} = b(0) + \lambda b(1) + f(0) + \lambda f(1),$$

$$\left\{\tilde{V}, \frac{\partial \tilde{U}}{\partial u_3}\right\} = c(0) + \lambda c(1), \qquad \left\{\tilde{V}, \frac{\partial \tilde{U}}{\partial u_4}\right\} = b(0) + \lambda b(1),$$

$$\left\{\tilde{V}, \frac{\partial \tilde{U}}{\partial \lambda}\right\} = 2(b(0) + \lambda b(1) + d(0) + \lambda d(1)),$$

where

$$\tilde{V} = \begin{pmatrix} a(0) + \lambda a(1) \\ b(0) + \lambda b(1) \\ c(0) + \lambda c(1) \\ d(0) + \lambda d(1) \\ f(0) + \lambda f(1) \\ g(0) + \lambda g(1) \end{pmatrix},$$

$$a(0)=\sum_{m\geq 0}^{\infty}a(0,m)\lambda^{-2m},\qquad a(1)=\sum_{m\geq 0}^{\infty}a(1,m)\lambda^{-2m},\ldots.$$

Substituting the above computing results into the trace variational identity

$$\frac{\delta}{\delta u_i}\int\left\{V,\frac{\partial U}{\partial\lambda}\right\}dx=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\left(\lambda^{\gamma}\left\{V,\frac{\partial U}{\partial u_i}\right\}\right),\quad i=1,2,\ldots,4$$

yields

$$\frac{\delta}{\delta u}\int[2(b(0)+\lambda b(1)+d(0)+\lambda d(1))]dx=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix}c(0)+\lambda c(1)+g(0)+\lambda g(1)\\ b(0)+\lambda b(1)+f(0)+\lambda f(1)\\ c(0)+\lambda c(1)\\ b(0)+\lambda b(1)\end{pmatrix}. \tag{17}$$

Comparing the coefficients of λ^{-2n-1} in (17) gives rise to

$$\frac{\delta}{\delta u}\int[2b(1,n+1)+2d(1,n+1)]dx=(-2n+\gamma)\begin{pmatrix}c(0,n)+g(0,n)\\ b(0,n)+f(0,n)\\ c(0,n)\\ b(0,n)\end{pmatrix}. \tag{18}$$

Again comparing the coefficients of λ^{-2n-2} in (17) yield

$$\frac{\delta}{\delta u}\int[2b(0,n+1)+2d(0,n+1)]dx=(-2n-1+\gamma)\begin{pmatrix}c(1,n+1)+g(1,n+1)\\ b(1,n+1)+f(1,n+1)\\ c(1,n+1)\\ b(1,n+1)\end{pmatrix}. \tag{19}$$

Inserting the initial values in (8) into (18), (19) gives $\gamma=0$. Thus, the relations (18), (19) can determine the following two Hamiltonian functions

$$\begin{cases}\dfrac{\delta H(1,2n+2)}{\delta u}=\begin{pmatrix}c(0,n)+g(0,n)\\ b(0,n)+f(0,n)\\ c(0,n)\\ b(0,n)\end{pmatrix},\\[2ex] H(1,2n+2)=-\int\dfrac{2b(1,n+1)+2d(1,n+1)}{2n}dx,\end{cases} \tag{20}$$

$$\begin{cases}\dfrac{\delta H(2,2n+1)}{\delta u}=\begin{pmatrix}c(1,n+1)+g(1,n+1)\\ b(1,n+1)+f(1,n+1)\\ c(1,n+1)\\ b(1,n+1)\end{pmatrix},\\[2ex] H(2,2n+1)=-\int\dfrac{2b(0,n+1)+2d(0,n+1)}{2n+1}dx.\end{cases} \tag{21}$$

Therefore, we have

$$u_t=J_1\frac{\delta H(2,2n+1)}{\delta u}=J_2\frac{\delta H(1,2n+2)}{\delta u}. \tag{22}$$

In terms of (8), a recurrence operator L is given by

$$\begin{pmatrix} c(1,n+1)+g(1,n+1)\\ b(1,n+1)+f(1,n+1)\\ c(1,n+1)\\ b(1,n+1)\end{pmatrix}=L\begin{pmatrix} c(0,n)+g(0,n)\\ b(0,n)+f(0,n)\\ c(0,n)\\ b(0,n)\end{pmatrix}=L^2\begin{pmatrix} c(1,n)+g(1,n)\\ b(1,n)+f(1,n)\\ c(1,n)\\ b(1,n)\end{pmatrix}$$

where

$$L=\begin{pmatrix} -\frac{1}{2}\partial+u_2\partial^{-1}u_1 & -u_2\partial^{-1}u_2 & u_2\partial^{-1}u_3+u_4\partial^{-1}u_1-p(x) & -u_2\partial^{-1}u_4-u_4\partial^{-1}u_2\\ u_1\partial^{-1}u_1 & \frac{1}{2}\partial-u_1\partial^{-1}u_2 & u_1\partial^{-1}u_3+u_3\partial^{-1}u_1 & -u_1\partial^{-1}u_4-u_3\partial^{-1}u_2-p(x)\\ 0 & 0 & -\frac{1}{2}\partial+u_2\partial^{-1}u_1 & -u_2\partial^{-1}u_2\\ 0 & 0 & u_1\partial^{-1}u_1 & \frac{1}{2}\partial-u_1\partial^{-1}u_2 \end{pmatrix}$$

which satisfies

$$J_1L=L^*J_1=J_2. \tag{23}$$

Which implies (22) is Liouville integrable hierarchy and L^* is hereditary.

In terms of the definition of integrable couplings, we conclude that the system (22) is the integrable couplings of the generalized AKNS hierarchy.

4 A Reduction of the Integrable Couplings

In what follows, we consider a reduction cases of the integrable couplings (22).

When $n=1$, the system (22) reduces to

$$\begin{cases} u_{1t}=\frac{1}{2}\beta(u_{1xx}-2u_1^2u_2),\\ u_{2t}=-\frac{1}{2}\beta(u_{2xx}-u_2^2u_1),\\ u_{3t}=\beta\left(\frac{1}{2}u_{3xx}-p_x(x)u_1-2p(x)u_{1x}-u_1^2u_4-2u_1u_2u_3\right),\\ u_{4t}=\beta\left(-\frac{1}{2}u_{4xx}-p_x(x)u_2-2p(x)u_{2x}+u_2^2u_3+2u_1u_2u_4\right). \end{cases} \tag{24}$$

Taking $u_2=\pm u_1^*$, $\beta=2i$, the above system (24) reduces to the integrable couplings of the Schrödinger equation as follows:

$$\begin{cases} iu_{1t}+u_{1xx}\mp 2u_1|u_1|^2=0,\\ iu_{3t}+u_{3xx}-2u_1p_x(x)-4p(x)u_{1x}-2u_1^2u_4\mp 4u_3|u_1|^2=0,\\ iu_{4t}-u_{4xx}\mp 2u_1^*p_x(x)\mp 4p(x)u_{1x}^*\pm 4u_4|u_1|^2+2u_1^{*2}u_3=0. \end{cases} \tag{25}$$

Here if we take $u_4=\pm u_3^*$, $p(x)=iq(x)$($q(x)$ is real function), the systems (25) becomes

$$\begin{cases} iu_{1t}+u_{1xx}\mp 2u_1|u_1|^2=0,\\ iu_{3t}+u_{3xx}-2iu_1q_x(x)-4iq(x)u_{1x}\mp 2u_1^2u_3^*\mp 4u_3|u_1|^2=0, \end{cases} \tag{26}$$

when we choose different $q(x)$ in (26), we can obtain many new integrable couplings of the Schrödinger equation. Especially, taking $q(x)=0$, we easily give the following integrable couplings of the Schrödinger equation

$$\begin{cases} iu_{1t} + u_{1xx} \mp 2u_1|u_1|^2 = 0, \\ iu_{3t} + u_{3xx} \mp 2u_1^2 u_3^* \mp 4u_3|u_1|^2 = 0. \end{cases} \tag{27}$$

5 Conserved Quantities

Let $\{H(2,2n+1)\}, \{H(1,2n+2)\}$ be defined by (20) and (21), then we have

$$\{H(2,2n+1), H(2,2m+1)\} = \left\{\frac{\delta H(2,2n+1)}{\delta u}, J_1 \frac{\delta H(2,2m+1)}{\delta u}\right\} = 0,$$

$$\{H(1,2n+2), H(1,2m+2)\} = \left\{\frac{\delta H(1,2n+2)}{\delta u}, J_1 \frac{\delta H(1,2m+2)}{\delta u}\right\} = 0.$$

Therefore, for the m-th integrable couplings (22), we can compute

$$\begin{aligned} \frac{d}{dt_m} H(2,2n+1) &= \int \frac{\delta H(2,2n+1)}{\delta u} u_{t_m} dx = \int \frac{\delta H(2,2n+1)}{\delta u} J_1 \frac{\delta H(2,2m+1)}{\delta u} dx \\ &= \{H(2,2n+1), H(2,2m+1)\} = 0, \\ \frac{d}{dt_m} H(1,2n+2) &= \int \frac{\delta H(1,2n+2)}{\delta u} u_{t_m} dx = \int \frac{\delta H(1,2n+2)}{\delta u} J_1 \frac{\delta H(1,2m+2)}{\delta u} dx \\ &= \{H(1,2n+2), H(1,2m+2)\} = 0. \end{aligned}$$

Which imply that the integrable couplings (22) possesses many conserved quantities: $\{H(2,2n+1)\}, \{H(1,2n+2)\}$.

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